

Tutorial 2: Numerical solution of ODEs

Predator-Prey model

In this tutorial we will numerically solve the predator-prey model of two competing species. The predator-prey model describes the interaction of two species in an ecosystem. In this example we will consider sharks as the predators competing with fish as their prey. If we consider a group of sharks and fish that don't interact with any other species but each other, then we can write the equations governing their populations as

$$\begin{aligned} \text{Rate of change of fish population} &= \text{Rate at which fish are born} \\ &\quad - \text{Rate at which fish are eaten by sharks,} \\ \text{Rate of change of shark population} &= \text{Rate at which fish are turned into sharks} \\ &\quad - \text{Rate at which sharks die without fish.} \end{aligned}$$

The first order system of ODEs governing the fish population F and the shark population S is then given by

$$\frac{dF}{dt} = \alpha F - \beta FS, \quad (1)$$

$$\frac{dS}{dt} = \epsilon \beta FS - \gamma S, \quad (2)$$

$$(3)$$

with initial conditions

$$\begin{aligned} F(0) &= F_0, \\ S(0) &= S_0, \end{aligned} \quad (4)$$

where F_0 and S_0 are the initial fish and shark populations, respectively. The parameters are given by

α : growth rate of fish in the absence of sharks (1/years).

γ : death rate of sharks in the absence of their prey, fish (1/years).

β : death rate per encounter of fish with sharks (1/sharks/years).

ϵ : efficiency of turning predated fish into sharks (sharks/fish).

From these equations we can see that in the absence of sharks, $S = 0$, and the fish population is given by

$$\frac{dF}{dt} = \alpha F, \quad (5)$$

so that the fish population grows exponentially according to

$$F(t) = F_0 e^{\alpha t}. \quad (6)$$

Likewise, in the absence of fish, $F = 0$, and the shark population is given by

$$\frac{dS}{dt} = -\gamma S, \quad (7)$$

so that the shark population decreases exponentially according to

$$S(t) = S_0 e^{-\gamma t}. \quad (8)$$

When the two species encounter each other the problem is more complicated. We will solve for the evolution of the sharks and fish numerically.

Nondimensionalization

Before solving any problem numerically, it is always important to first nondimensionalize the problem, so that the equations you are solving do not have any units. As they are, equations (1) and (2) are in units of population/year. The idea behind nondimensionalization is to simplify the governing equations so that the number of undetermined constants in them is a minimum.

If we nondimensionalize according to

$$\begin{aligned} F^* &= \frac{F}{F_0}, \\ S^* &= \frac{S}{S_0}, \\ t^* &= \alpha t, \end{aligned} \quad (9)$$

where F_0 and S_0 are the initial fish and shark populations and α is the growth rate of the fish, and substitute into equations (1) and (2), then we have

$$\begin{aligned} \alpha \frac{dF^* F_0}{dt^*} &= \alpha F^* F_0 - \beta F^* F_0 S^* S_0, \\ \alpha \frac{dS^* S_0}{dt^*} &= \epsilon \beta F^* F_0 S^* S_0 - \gamma S^* S_0. \end{aligned}$$

The nondimensional form of the governing ODEs is then given by

$$\frac{dF^*}{dt^*} = F^* - \frac{\beta S_0}{\alpha} F^* S^*, \quad (10)$$

$$\frac{dS^*}{dt^*} = \frac{\epsilon \beta F_0}{\alpha} F^* S^* - \frac{\gamma}{\alpha} S^*. \quad (11)$$

So we see that nondimensionalizing reduces the number of undetermined constants in the problem from 4 to 3, since the above system of ODEs can be written as

$$\frac{dF^*}{dt^*} = F^* - aF^*S^*, \quad (12)$$

$$\frac{dS^*}{dt^*} = bF^*S^* - cS^*, \quad (13)$$

with initial conditions

$$\begin{aligned} F^*(0) &= 1, \\ S^*(0) &= 1. \end{aligned}$$

Rather than continue to carry around the cumbersome $*$, in practice we rewrite the nondimensional equations without them but remember that the new equations are in **nondimensional** form,

$$\frac{dF}{dt} = F - aFS, \quad (14)$$

$$\frac{dS}{dt} = bFS - cS, \quad (15)$$

with initial conditions

$$\begin{aligned} F(0) &= 1, \\ S(0) &= 1, \end{aligned}$$

where the new constants are given by

$$\begin{aligned} a &= \frac{\beta S_0}{\alpha}, \\ b &= \frac{\epsilon\beta F_0}{\alpha}, \\ c &= \frac{\gamma}{\alpha}. \end{aligned}$$

Vector form

To solve the predator-prey model numerically, we must first write it in vector form. If we let $y_1 = F$ and $y_2 = S$, then equations (14) and (15) become

$$\frac{dy_1}{dt} = y_1 - ay_1y_2, \quad (16)$$

$$\frac{dy_2}{dt} = by_1y_2 - cy_2, \quad (17)$$

with initial conditions $y_1(0) = 1$ and $y_2(0) = 1$. In matrix-vector form this system is given by

$$\frac{d\mathbf{y}}{dt} = \mathbf{F}(t, \mathbf{y}), \quad (18)$$

where

$$\mathbf{F} = \begin{pmatrix} y_1 - ay_1y_2 \\ by_1y_2 - cy_2 \end{pmatrix}, \quad (19)$$

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad (20)$$

with initial condition

$$\mathbf{y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (21)$$

Problem specifics

The equations above govern the fish and shark populations. Let's say that we would like to determine what the maximum and minimum fish and shark populations will be over the course of 50 years if the initial fish and shark populations are $F_0 = 200$ and $S_0 = 50$, respectively. In order to determine this, we need actual numbers for the constants. These have been determined experimentally by studying populations in nature. Let's say for our problem, they are given by

$$\begin{aligned} \alpha &= 0.7 \text{ per year,} \\ \gamma &= 0.5 \text{ per year,} \\ \beta &= 0.007 \text{ per shark per year,} \\ \epsilon &= 0.1 \text{ shark per fish.} \end{aligned}$$

From this we can determine that the nondimensional constants are given by

$$\begin{aligned} a &= 0.5, \\ b &= 0.6, \\ c &= 0.7143. \end{aligned}$$

We also have to nondimensionalize the total time of 50 years with the original nondimensionalization using $t^* = \alpha t = 0.7 \text{ per year} \times 50 \text{ year} = 35$. Therefore, we need to compute the nondimensional ODE for 35 nondimensional time units to determine the maximum and minimum fish and shark populations over that period.

Numerical method

Now that we have a nondimensional ODE to solve and the exact values we would like to determine (maximum and minimum populations), we need to decide on the method we will use to solve it. In this example, we will use the forward Euler method. For a given system of ODEs

$$\frac{d\mathbf{y}}{dt} = \mathbf{F}(t, \mathbf{y}), \quad (22)$$

the Euler method updates the new value of \mathbf{y} with the forward derivative to obtain

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \Delta t \mathbf{F}_n. \quad (23)$$

Remember, however, that this method is very inaccurate since it is only a first order method.

Having chosen the method, we need to determine what time step Δt we will use. The time step is chosen either for stability reasons or for accuracy reasons, or both. Since we have not covered stability yet, we will choose the time step with accuracy in mind. Let's just say for now that we want to have a time step of $\Delta t = 1$ week. In terms of years, this is given by $\Delta t = 0.0192$ years. In nondimensional terms, this is given by $\Delta t^* = \alpha \Delta t = 0.7 \text{ per year} \times 0.0192 \text{ years} = 0.0134$. We will do some convergence analyses to see how accurate our answer is based on the time step we have chosen.

Problem summary

Given the nondimensional system of ODEs

$$\begin{aligned}\frac{dy_1}{dt} &= y_1 - ay_1y_2, \\ \frac{dy_2}{dt} &= by_1y_2 - cy_2,\end{aligned}$$

with constants

$$\begin{aligned}a &= 0.5, \\ b &= 0.6, \\ c &= 0.7143,\end{aligned}$$

and initial conditions

$$\begin{aligned}y_1(0) &= 1, \\ y_2(0) &= 1,\end{aligned}$$

compute the dynamics of y_1 and y_2 for $0 < t < 35$ using the forward Euler method with a time step of $\Delta t = 0.0134$, and compute the maximum and minimum values of y_1 and y_2 over that time period.

Pseudocode

Before proceeding to write the computer program, we need to be sure we know how we will implement the Euler method on the computer. This is done by writing the Euler method down in pseudocode first as follows:

Set the initial values $y_1(1) = 1$ and $y_2(1) = 1$.

Determine the number of time steps $n_{\max} = t_{\max} / \Delta t$

For $n=1$ to n_{\max}

Determine the right hand side \mathbf{F}_n

$$f_1 = y_1(n) - ay_1(n)y_2(n)$$

$$f_2 = by_1(n)y_2(n) - cy_2(n)$$

```
Update the values
y1(n + 1) = y1(n) + Δt f1
y2(n + 1) = y2(n) + Δt f2
end for
```

Computer code

This will be your task for this tutorial. You need to find the maximum and minimum values of the shark and fish populations by writing an m-file that computes the evolution of the shark and fish populations in **Octave**. Once you have computed the evolution of the populations over the specified time, you can determine the maximum and minimum populations over the desired interval.

The m-file can be found at:

http://fluid.stanford.edu/~fringer/courses/uwc/downloads/pred_prey_euler.m

Answer

The nondimensional fish and shark populations as a function of the nondimensional time are shown in Figure 1. From the data shown in this figure we can compute the minimum and

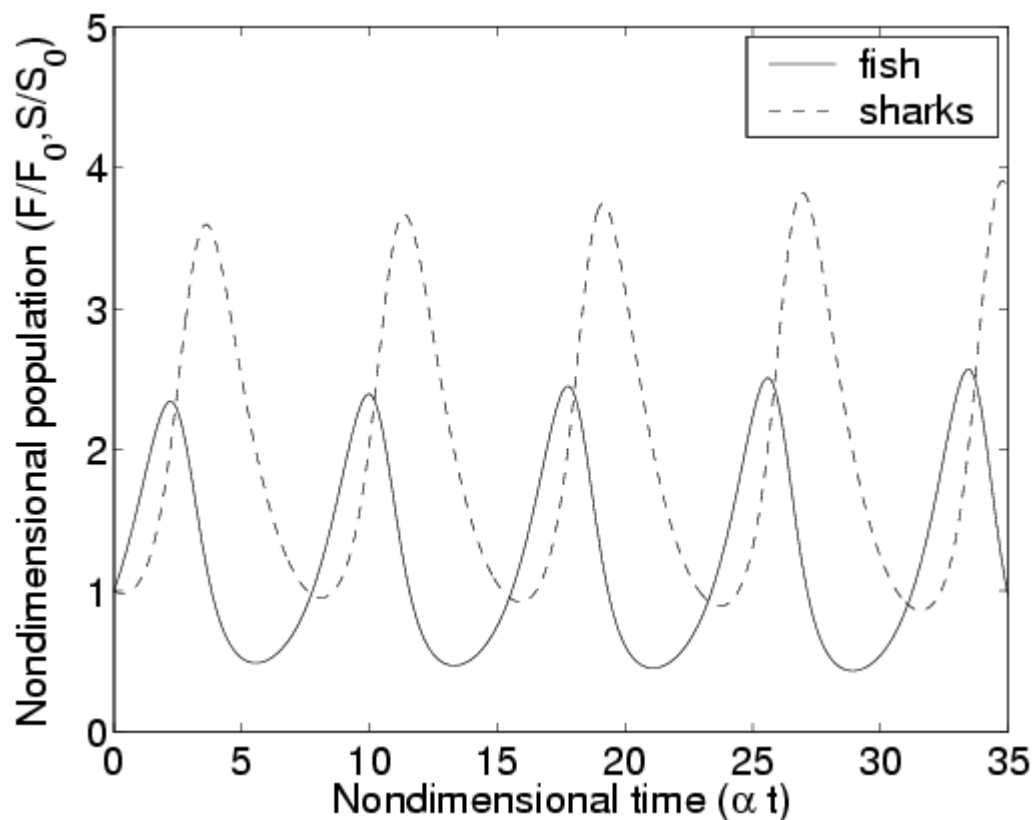


Figure 1: Nondimensional fish and shark populations as a function of nondimensional time.

Minimum fish population	86
Maximum fish population	514
Minimum shark population	43
Maximum shark population	195

maximum populations of each species to obtain (in dimensional form):

How confident can we be with this result? Assuming that the governing equations determine the exact behavior, we need to determine how much our answer will vary if we change the time step in the calculation. This is shown in the following table. The first row shows the result using a time step of 1 week. The next rows show the result when the time step is half the previous time step. We can see that using a time step of 1 week does not give us

Time step (weeks)	F_{min}	F_{max}	S_{min}	S_{max}
1.00e+00	86	514	43	195
5.00e-01	94	488	46	186
2.50e-01	97	477	48	182
1.25e-01	99	471	48	180
6.25e-02	100	468	49	179
3.13e-02	100	467	49	178
1.56e-02	100	466	49	178
7.81e-03	100	466	49	178

very good results, because halving the time step to 0.5 week changes the answers by quite a bit. But as we continue to decrease the time step, we can see that we are converging on the result given in the table below. In order to improve our result so that we know we are only

Minimum fish population	100
Maximum fish population	466
Minimum shark population	49
Maximum shark population	178

off by 1 in the computations of the minimum and maximum populations, we had to compute the population histories with a total of 128 times as many time steps as we originally began with when we used a time step of 1 week. This is because the Euler method only reduces the error by one half each time we half the time step. For your homework you will compute the minimum and maximum values of the populations with higher order methods, and you will see the advantages of using higher order methods because they converge upon the solution much more rapidly.

You can download the m-file used to obtain the convergence results at

http://fluid.stanford.edu/~fringer/courses/uwc/downloads/pred_pre_euler_converge.m