Assignment 4: Numerical solution of boundary value problems

Due: 4 September 2002, by 5:00 pm

In this assignment you will be solving the second order nonlinear ODE

$$y'' - 2yy' = x \tag{1}$$

subject to the boundary conditions $y(0) = 1$ and $y(1) = -1$. Use $N = 20$ points to discretize this boundary value problem for all of the questions in this assignment.

Problem 1

Write the boundary value problem in the form

$$\frac{d\mathbf{y}}{dx} = \mathbf{f}(x, \mathbf{y}) \,, \tag{2}$$

where

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix} \,, \tag{3}$$

and determine the form of $\mathbf{f}(x, \mathbf{y})$ and the boundary conditions $\mathbf{y}(0)$ and $\mathbf{y}(1)$. Leave question marks where you do not know the conditions a-priori.

Problem 2

Write an Octave function called "guess" that shoots from $x = 0$ to $x = 1$ with the Euler predictor-corrector method with starting values $y_1(0) = y_{10}$ and $y_2(0) = y_{2g}$, where y_{10} is the initial value of $y_1(0)$, and y_{2g} is the guessed value of $y_2(0)$. When you shoot from $x = 0$ to $x = 1$ with a guessed value of y_{2g}, in general, you will not obtain the correct value of $y_1(1) = -1$, which is specified as a boundary condition, but rather, you will obtain a guessed value, which we will call y_{1g}. Your Octave function should take as its inputs the starting values of $y_1(0)$ and $y_2(0)$, as well as the boundary condition $y_1(1) = -1$, and it should return the result of the shooting method $y_1(x)$ and $y_2(x)$ as well as the error in computing the value of $y_1(1)$. Your function should be used in the form

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>> [E,y1,y2]=guess(x,y10,y2g,y11);
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where $x = [0 : dx : 1]$ is the discretized grid with $dx = 1/(N-1)$ and N is the number of grid points, $y10$ is the given value of $y_1(0)$, $y2g$ is the guessed value of $y_2(0)$, and $y11$ is the desired value $y_1(1)$. The returned values $y1$ and $y2$ are the solutions of $y_1(x) = y(x)$ and $y_2(x) = y'(x)$ obtained with the shooting method, and $E = y1g - y11$ is the error resulting

from the guess of the boundary value $y_1(1)$. Using $N = 20$ points, plot the result $y_1(x)$ and $y_2(x)$ when the shooting method is used with a guessed value of $y_{2g} = 0$, and determine the value of y_{1g}.

Hint: Download the following code to help you with the implementation of the guess function:

`http://fluid.stanford.edu/~fringer/courses/uwc/downloads/assign4prob2.m`

Problem 3

Using your guess function from Problem 2, plot the error E in the estimation of $y_1(1)$ as a function of the initial guessed values of $y_{2g} = -2.5, -2.25, -2, -1.75, -1.5, -1.25$, and -1.

Problem 4

Use the two points in the plot of Problem 3 that bracket the root where $E = 0$ to approximate the correct value of y_{2g} with the secant method. Using this value as the starting value for y_{2g} in the guess function, obtain and plot the approximate solution of the boundary value problem $y_1(x)$ and $y_2(x)$ as functions of x.

Extra credit (can be done instead of Problems 3 and 4)

Obtain the value of y_{2g} using the secant method with starting values of $s_1 = 1.0$ and $s_2 = 0.5$ (see Lecture 9 for details) and continue to iterate until $E(s) < 10^{-7}$. Using this converged value as the starting value for y_{2g} in the guess function, obtain and plot the approximate solution of the boundary value problem $y_1(x)$ and $y_2(x)$ as functions of x.